



Absorbing boundary conditions for simulation of gravitational waves with spectral methods in spherical coordinates

Jérôme Novak ^{*}, Silvano Bonazzola

Laboratoire de l'Univers et de ses Théories (UMR8102 du C.N.R.S.), Observatoire de Paris, Section de Meudon, F-92195 Meudon Cedex, France

Received 3 April 2002; received in revised form 4 September 2003; accepted 26 November 2003
Available online 23 December 2003

Abstract

We present a new formulation of the multipolar expansion of an exact boundary condition for the wave equation, which is truncated at the quadrupolar order. Using an auxiliary function, that is the solution of a wave equation on the sphere defining the outer boundary of the numerical grid, the absorbing boundary condition is simply written as a perturbation of the usual Sommerfeld radiation boundary condition. It is very easily implemented using spectral methods in spherical coordinates. Numerical tests of the method show that very good accuracy can be achieved and that this boundary condition has the same efficiency for dipolar and quadrupolar waves as the usual Sommerfeld boundary condition for monopolar ones. This is of particular importance for the simulation of gravitational waves, which have dominant quadrupolar terms, in General Relativity.

© 2003 Elsevier Inc. All rights reserved.

Keywords: Absorbing boundary conditions; Spectral methods; Wave equation; General relativity

1. Introduction

1.1. Wave equations in general relativity

The determination of numerical solutions of the Einstein equations is the scope of *numerical relativity*. It is a fundamental issue not only for the determination of gravitational wave signals for detector data analysis, but also for the study of the properties of relativistic astrophysical objects [1]. Within numerical relativity studies, the most commonly used formulation of the Einstein equations is the so-called “3 + 1” formalism (also called *Cauchy formalism* [2]) in which space–time is foliated by a family of space-like hypersurfaces Σ_t , which are described by their 3-metric γ_{ij} . The 4-metric $g_{\mu\nu}$ is then described in terms of γ_{ij} ,

^{*} Corresponding author.

E-mail addresses: Jerome.Novak@obspm.fr (J. Novak), Silvano.Bonazzola@obspm.fr (S. Bonazzola).

a 3-vector N^i (called *shift*) and a scalar N (called *lapse*). In this formalism, the Einstein equations can be decomposed into a set of four constraint equations and six second-order dynamical equations. Solving the Einstein equations then turns to be a Cauchy problem of evolution under constraints and there remains the freedom to choose the time coordinate (slicing) and the spatial gauge.

For example, the choice of *maximal slicing* for the time coordinate (see [3]) converts the constraint equations to scalar form and a vectorial Poisson-like equation, for which a numerical method for solution has been presented in [4]. As far as evolution equations are concerned, they consist of six non-linear scalar wave equations in curved space–time, with the additional choice of the *Dirac* gauge [3]. The whole system is a mixed initial value-boundary problem, and this paper deals with boundary conditions for the time evolution equations. Indeed, a simpler problem is considered: the initial value-boundary problem for a linear and flat scalar wave equation:

$$\square\phi(t, r, \theta, \varphi) = \sigma(t, r, \theta, \varphi), \tag{1}$$

where

$$\square\phi = \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial r^2} - \frac{2}{r} \frac{\partial\phi}{\partial r} - \frac{1}{r^2} \left(\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\tan\theta} \frac{\partial\phi}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2\phi}{\partial\varphi^2} \right)$$

is the usual flat scalar d’Alembert operator in spherical coordinates (r, θ, φ) and σ is a source. To solve a more general problem in curved space–time, like for example:

$$\frac{\partial^2\phi}{\partial t^2} - \mu^2(t, r)\Delta\phi = \phi^2, \tag{2}$$

one can put non-linear terms to the source σ and represent at each time-step the metric function μ^2 by a polynomial (semi-implicit scheme, see [5] for an example in spherical symmetry).

1.2. Motivations for general quadrupolar absorbing boundary conditions

The study of the simple wave equation and its properties concerning quadrupolar waves is more than a toy-model for numerical relativity. There are many degrees of freedom in the formulation of the Einstein equations and in the gauge choice. It is not clear which of these formulations are well-posed or numerically stable [6]. It is therefore important to have numerical tools that are general in the sense that they can be used within the framework of various formulations and gauges. Still, in many cases, the dynamical degrees of freedom of the gravitational field can be described by wave-like propagation equations in curved space–time. On the other hand, since we are mainly interested in the gravitational wave signal, which has a quadrupolar dominant term, we have to make high precision numerical models (including boundary conditions) to study this mode, as well as lower multipoles.

These statements can be illustrated as follows. One of the main sources we want to study are binaries of two compact objects (neutron star or black hole) orbiting around each other. Gravitational waves take away angular momentum and the system coalesces. In some perturbative approach, the terms corresponding to this “braking force” result from a subtle cancellation between terms of much higher amplitude [7]. In numerical non-perturbative studies, the same phenomenon may happen and, if the dominant modes of the wave are not computed with enough precision, the angular momentum loss may be strongly over-estimated. Moreover, the time-scale for coalescence is much larger than the orbital period and the system is almost stationary.

There has been many interesting developments concerning absorbing boundaries in the last years, with the Perfectly Matched Layers (PML, see [8,9]) which consist in surrounding the true domain of interest by an absorbing layer where the wave is damped. These methods may not be the best-suited for our problems

since, as stated above, we might have to change the formulation of the equations we want to solve. Moreover, the main problem we want to address is the simulation of quadrupolar waves and, as it will be shown later in this paper, with our formulation it is possible to have a clear control on the behavior of these quadrupolar waves. Finally, this formulation is straightforward to implement and very little CPU time consuming in the context of spectral methods and spherical coordinates, which we are already using to solve elliptic partial differential equations (PDE) arising in numerical relativity (scalar and vectorial ones, see [4]). The development and implementation of the PML techniques for our problem would require much more work and computing time, whereas it is not guaranteed at all it would give better results. For all these reasons we chose to develop a new formulation of the Bayliss and Turkel [10] boundary conditions, particularly well suited for using with spectral methods and spherical coordinates.

The paper deals with this new formulation as well as numerical tests. It is organized as follows. First, Section 2 presents boundary conditions: it briefly recalls main results from Bayliss and Turkel (2.1) and we then derive the formulation adapted up to quadrupolar modes of the wave (2.2). Then, Section 3 briefly describes spectral methods in spherical coordinates that were used (3.1) and details the numerical results (3.2). Finally, Section 4 gives a summary and some concluding remarks.

2. Absorbing boundary conditions

An important difference between the solution of the wave equation and that of the Poisson equation (as in [4]) is the fact that boundary conditions cannot be imposed at infinity, since one cannot use “compactification”, i.e. a change of variable of the type $u = 1/r$. This type of compactification is not compatible with an hyperbolic PDE, see [11]. One has to construct an artificial boundary and impose conditions on this surface to simulate an infinite domain. These conditions should therefore give no reflection of the wave, that could spuriously act on the evolution of the system studied inside the numerical grid. The boundary conditions have to *absorb* all the waves that are coming to the outer limit of the grid. The general *condition of radiation* is derived e.g. in [11], and defined as

$$\lim_{r \rightarrow \infty} \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t} \right) (r\phi) = 0. \quad (3)$$

At a finite distance $r = R$ the condition, which is then approximate, reads

$$\left. \frac{\partial \phi}{\partial t} + \frac{\phi}{R} + \frac{\partial \phi}{\partial r} \right|_{r=R} = 0, \quad (4)$$

which will be hereafter referred as the “Sommerfeld condition” and is exact only for pure monopolar waves. A completely general and exact boundary condition for the wave equation on an artificial spherical boundary has recently been derived by Aladl et al. [12] and involves an infinite series of inverse Fourier transforms of the solution. This condition may not be suitable for direct numerical implementation for which Aladl et al. derived a truncated approximate condition.

2.1. Asymptotic expansion in terms of multipolar momenta

A rather general method to impose non-reflecting boundary conditions is to construct a sequence of boundary conditions that, for each new term, are in some sense giving better results. Some of the possibilities to define “better” are when the reflected wave decreases:

- as the incident wave approaches in a direction closer to some preferred direction(s) (see e.g. [13]),
- for shorter wavelengths,
- as the position of artificial boundary goes to infinity.

This last approach is the most relevant to the problem of solving the Einstein equation for isolated systems. It is also a way of expanding condition (3) in terms of asymptotic series, which has been studied in [10], where a sequence of recursive boundary conditions is derived. Let us recall here some of their results.

A radiating solution of (1) with the source $\sigma = 0$ can be written as the following expansion:

$$\phi(t, r, \theta, \varphi) = \sum_{k=1}^{\infty} \frac{f_k(t-r, \theta, \varphi)}{r^k}. \tag{5}$$

The operators acting on a function $f(t, r, \theta, \varphi)$ are recursively defined by:

$$B_1 f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} + \frac{f}{r}, \tag{6}$$

$$B_{n+1} f = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{2n+1}{r} \right) B_n f. \tag{7}$$

The family of boundary conditions then reads:

$$B_n \phi|_{r=R} = 0. \tag{8}$$

In [10], it is shown that, following from (5), a radiating solution of the wave equation verifies:

$$B_n \phi = O\left(\frac{1}{r^{2n+1}}\right), \tag{9}$$

which in particular means that condition (8) is an asymptotic one in powers of $1/r$. The condition $B_1 \phi = 0$ is the same as the Sommerfeld condition (4) and the same as the first approximation in terms of the angle between the direction of propagation of the wave and the normal to the boundary, derived in [13].

Finally, using expression (5) one can verify that the operator B_n annihilates the first n terms of the expansion. Thinking in terms of spherical harmonics, this means that condition (8) is exact if the wave carries only terms with $l \leq n - 1$. In other words, the reflection coefficients for all modes lower than n are zero. Since we are interested in the study of gravitational wave emission by isolated systems, it is of great importance to have a very accurate description of the quadrupolar part of the waves, which is dominant. Therefore, if the $l = 2$ part of the gravitational wave is well described, higher-order terms may not play such an important role in the dynamical evolution of the system. The situation then is not so bad even if only an approximate boundary condition is imposed for those terms with $l \geq 3$. Moreover, the error on the function scales like $1/R^{n+1}$ so, if we impose

$$B_3 \phi|_{r=R} = 0, \tag{10}$$

we have an exact boundary condition for the main contribution to the gravitational wave and an error going to zero as $O(1/R^4)$. When developing this expression, one gets:

$$\left(\frac{\partial^3}{\partial t^3} + 3 \frac{\partial^3}{\partial t^2 \partial r} + 9 \frac{1}{r} \frac{\partial^2}{\partial t^2} + 3 \frac{\partial^3}{\partial t \partial r^2} + 18 \frac{1}{r^2} \frac{\partial}{\partial t} + 18 \frac{1}{r} \frac{\partial^2}{\partial t \partial r} + \frac{\partial^3}{\partial r^3} + 9 \frac{1}{r} \frac{\partial^2}{\partial r^2} + 18 \frac{1}{r^2} \frac{\partial}{\partial r} + 6 \frac{1}{r^3} \right) \phi \Big|_{r=R} = 0. \tag{11}$$

2.2. New formulation for quadrupolar terms

Starting from (11) and considering that ϕ is a solution of Eq. (1), we replace second radial derivatives with:

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{\partial^2 \phi}{\partial t^2} - \frac{2}{r} \frac{\partial \phi}{\partial r} - \frac{1}{r^2} \Delta_{\text{ang}} \phi, \quad (12)$$

where

$$\Delta_{\text{ang}} \phi = \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial \phi}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} \quad (13)$$

is the angular part of the Laplace operator. We are making here the assumption that, at the outer boundary of the grid ($r = R$), the source term σ of (1) is negligible. This is a very good approximation for our studies of isolated systems and is also the assumption made when writing a solution to the wave equation in the form (5). For example, the third-order radial derivative is replaced with

$$\frac{\partial^3 \phi}{\partial r^3} = \frac{\partial^3 \phi}{\partial t^2 \partial r} + \frac{2}{r^3} \Delta_{\text{ang}} \phi - \frac{1}{r^2} \Delta_{\text{ang}} \frac{\partial \phi}{\partial r} + \frac{2}{r^2} \frac{\partial \phi}{\partial r} - \frac{2}{r} \frac{\partial^2 \phi}{\partial r^2} \quad (14)$$

and the second-order radial derivatives of the last term (combined with its counterpart term in (11)) is replaced once more using (12). The boundary condition is then written as:

$$B_3 \phi = \left(4 \frac{\partial^3}{\partial t^3} + 4 \frac{\partial^3}{\partial t^2 \partial r} + 16 \frac{1}{r} \frac{\partial^2}{\partial t^2} + 18 \frac{1}{r^2} \frac{\partial}{\partial t} + 12 \frac{1}{r} \frac{\partial^2}{\partial t \partial r} + 6 \frac{1}{r^2} \frac{\partial}{\partial r} - \frac{3}{r^2} \Delta_{\text{ang}} \frac{\partial}{\partial t} - \frac{1}{r^2} \Delta_{\text{ang}} \frac{\partial}{\partial r} - \frac{5}{r^3} \Delta_{\text{ang}} + \frac{6}{r^3} \right) \phi. \quad (15)$$

We use the auxiliary function ξ :

$$\forall (t, \theta, \varphi), \quad B_1 \phi|_{r=R} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r} \right) \phi(t, r, \theta, \varphi) \Big|_{r=R} = \xi(t, \theta, \varphi), \quad (16)$$

which is defined on the sphere at $r = R$. Inserting this definition into the boundary condition $B_3 \phi|_{r=R} = 0$, with Eq. (15), one gets:

$$\frac{\partial^2 \xi}{\partial t^2} - \frac{3}{4R^2} \Delta_{\text{ang}} \xi + \frac{3}{R} \frac{\partial \xi}{\partial t} + \frac{3\xi}{2R^2} = \frac{1}{2R^2} \Delta_{\text{ang}} \left(\frac{\phi}{R} - \frac{\partial \phi}{\partial r} \Big|_{r=R} \right), \quad (17)$$

which is a wave-like equation on the outer boundary of the grid, with some source term, equal to zero if the solution ϕ is spherically symmetric. The boundary condition (10) is now equivalent to the system (16) and (17). Written in this way, this formulation can be regarded as a perturbation of the Sommerfeld boundary condition ($B_1 \phi = 0$) given by (16). The main advantages are that it can be very easily implemented using spectral methods and spherical coordinates (see Section 3.1) and that mixed derivatives have almost disappeared: there is only one remaining as a source of (17).

3. Numerical experiments

3.1. Implementation using spectral methods and spherical coordinates

Spectral methods ([14,15], for a review see [16]) are a very powerful approach for the solution of a PDE and, in particular, they are able to represent functions and their spatial derivatives with very high accuracy. As presented in [17], we decompose scalar fields on spherical harmonics $Y_l^m(\theta, \varphi)$, for the angular part:

$$\phi(t, r, \theta, \varphi) = \sum_{l=0}^L \sum_{m=-l}^l \phi_{lm}(t, r) Y_l^m(\theta, \varphi) \tag{18}$$

and on even Chebyshev polynomials ($T_{2k}(x = r/R)$) for the radial part of each $\phi_{lm}(t, r)$. Time derivatives are evaluated using finite-difference methods. Since Chebyshev collocation points are spaced by a distance of order $1/N^2$, (where N is the highest degree of the Chebyshev polynomials used for the radial decomposition) near grid boundaries, the Courant condition on the time step for explicit integration schemes of Eq. (1) also varies like $1/N^2$. This condition is very restrictive and it is therefore necessary to use an implicit scheme. We use the Crank–Nicholson scheme, which is unconditionally stable, as shown by various authors (see e.g. [14]). This scheme is second-order in time and the smoothing of the solution due to implicit time-stepping remains lower than the other errors discussed hereafter. This implicit scheme results in a boundary-value problem for ϕ at each time-step. The solution to this problem is obtained by inverting the resulting spatial operator acting on ϕ using the tau method. Its matrix (in Chebyshev coefficient space) has a condition number that is rapidly increasing with N . This can be alleviated by the use of preconditioning matrices, obtained from finite-differences operators (see [15]).

At the beginning of time integration, we suppose that ϕ satisfies the Sommerfeld boundary condition (4), that is $\forall(\theta, \varphi) \xi(t = 0, \theta, \varphi) = 0$. ξ is then calculated at next time-step using (17). This is done very easily since the angular parts of ϕ and ξ are decomposed on the basis of spherical harmonics; each component $\xi_{lm}(t)$ is the solution of a simple ODE in time, which is integrated using the same Crank–Nicholson scheme as for Eq. (1), with boundary conditions such that ξ is periodic on the sphere. This is already verified by the Y_l^m (Galerkin method). We get, with δt being the time-step, $\phi_{lm}^J(r) = \phi(J\delta t, r)$ and $\xi_{lm}^J = \xi_{lm}(J\delta t)$:

$$\begin{aligned} & \frac{\xi_{lm}^{J+1} - 2\xi_{lm}^J + \xi_{lm}^{J-1}}{\delta t^2} + \frac{3}{8} \frac{l(l+1)}{R^2} (\xi_{lm}^{J+1} + \xi_{lm}^{J-1}) + \frac{3}{R} \frac{\xi_{lm}^{J+1} - \xi_{lm}^{J-1}}{2\delta t} + \frac{3}{4R^2} (\xi_{lm}^{J+1} + \xi_{lm}^{J-1}) \\ & = \frac{l(l+1)}{2R^2} \left(\frac{\phi_{lm}^J(R)}{R} - \frac{\partial \phi_{lm}^J}{\partial r} \Big|_{r=R} \right). \end{aligned}$$

This equation in ξ_{lm}^{J+1} is solved and, for each pair (l, m) , we impose for ϕ_{lm}^{J+1}

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{r} \right) \phi_{lm}^{J+1}(r) \Big|_{r=R} = \xi_{lm}^{J+1},$$

which looks like a modification of the condition (4).

3.2. Tests on outgoing waves

The Sommerfeld boundary condition (4) is an exact condition, even at finite distance from the source, when only considering monopolar waves. In order to test our implementation of absorbing boundary condition (8), we compared its efficiency in being transparent to waves carrying only monopolar, dipolar and quadrupolar terms, to the efficiency of the Sommerfeld boundary condition for monopolar waves. We started with $\phi = 0$ at $t = 0$ and then solved Eq. (1) with

$$\sigma(t, r, \theta, \varphi) = \begin{cases} S(r, \theta, \varphi) e^{-1/r^2} e^{-1/(t-1)^2}, & 0 \leq t \leq 1, \\ \sigma(t, r, \theta, \varphi) = 0, & \text{otherwise} \end{cases} \tag{19}$$

with $S(r, \theta, \varphi)$ null for $r > R$.

In all cases, we performed a first calculation with a very large grid (considered as infinite, we checked with various values of the radius that the result in the interval $0 \leq r \leq R$ would be the same), so that in the time interval $[0, 2R + 1]$ the wave would not reach the boundary, on which we imposed an homogeneous

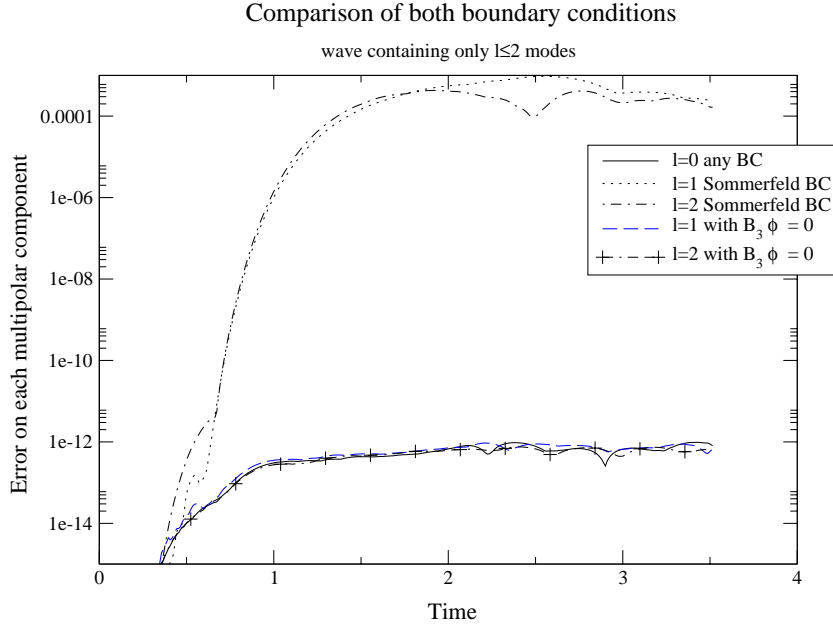


Fig. 1. Comparison between the efficiencies of $B_1\phi = 0$ (4) and $B_3\phi = 0$ (16) for $l \leq 2$ modes. The source of the wave equation is defined in Eqs. (19) and (20). We took $R = 1.2$, a time-step $\delta t = 10^{-6}$, 33 polynomials for radial decomposition, 5 for θ and 4 for φ .

boundary condition.¹ This gave us the reference solution crossing the $r = R$ sphere without any reflection. We then solved again the same problem, but on a grid of radius R , imposing Sommerfeld boundary conditions $B_1\phi = 0$ (4), or our quadrupolar boundary conditions $B_3\phi = 0$ through the system (16) and (17). The L_1 norm of the relative difference between the functions obtained on the small grid and the reference solution was taken as the error.

3.2.1. $l \leq 2$ case

First, we took

$$S(r, \theta, \varphi) = (e^{-r^2} - e^{-R^2})(r^2 \cos^2 \theta + r \sin \theta \cos \varphi), \quad (20)$$

which contains only $l \leq 2$ modes. Fig. 1 shows the relative efficiency of $B_3\phi = 0$ (16) condition to $B_1\phi = 0$ (4) for all three modes present in the wave generated by (20). For the monopolar ($l = 0$) mode, the evolution of the error would be the same for both types of boundary conditions, within 1% of difference on the error. As far as the discrepancy for dipolar and quadrupolar modes is concerned, one can see that it drops from 10^{-4} with Sommerfeld boundary condition, to 10^{-12} with $B_3\phi = 0$ (16). This lower level is the same as for the monopolar mode with the Sommerfeld boundary condition. We have checked that all solutions had converged with respect to the number of spectral coefficients and to the time-step. The error level at 10^{-12} is then mainly due to the condition number of the matrix operator we invert (see Section 3.1). We here conclude that our formulation of $B_3\phi = 0$ (16) is as efficient for waves containing only $l \leq 2$ modes as the Sommerfeld boundary condition (4) for monopolar waves.

¹ Results obtained here did not depend on the nature of boundary conditions.

3.2.2. Waves containing higher multipoles

The study has been extended to a more general source σ which contains a priori all multipolar terms:

$$S(r, \theta, \varphi) = (e^{-r^2} - e^{-R^2}) \left(e^{-4(x-0.7)^2} + e^{-3(x+0.5)^2} \right). \tag{21}$$

Of course, in numerical implementation, only a finite number of these terms are represented. The geometry of this source can be related to the distribution of mass in the case of a binary system of gravitating bodies, which is one of the main astrophysical sources of gravitational radiation we try to model. Let us make a comparison between the errors obtained, on the one hand with the condition $B_1\phi = 0$ (Fig. 2), and on the other hand with $B_3\phi = 0$ (Fig. 3).

As in the case in Fig. 1, the error in the monopolar component remains roughly the same, regardless of whether one uses boundary condition (4) or (16). The errors for the dipolar and quadrupolar components also exhibit similar properties: the use of condition (16) causes these errors to be of the same magnitude as the error in the monopolar term. In the case of Fig. 3, this level is higher than on Fig. 1 because a longer time-step has been used. Finally, we have also plotted the discrepancies between the reference and test solutions for the $l = 3$ multipole. Following [10], the boundary condition $B_3\phi = 0$ is not exact for this component. Nevertheless, one can see a reduction in the error for this component. This can be understood using the result of [10] which shows that the condition $B_3\phi = 0$ cancels the first 3 terms in the asymptotic development in powers of $1/r$ of the solution ϕ (9). Then, since a given multipolar term l_0 is present in terms like $1/r^n$ with $n \leq l_0$ (see e.g. [11]), it is clear that the condition $B_3\phi = 0$ is supposed to cancel all terms decaying slower than $1/r^4$ in the $l \geq 3$ mode. Thus, the error displayed in Fig. 4 is three orders of magnitude lower with the condition $B_3\phi = 0$ than with $B_1\phi = 0$.

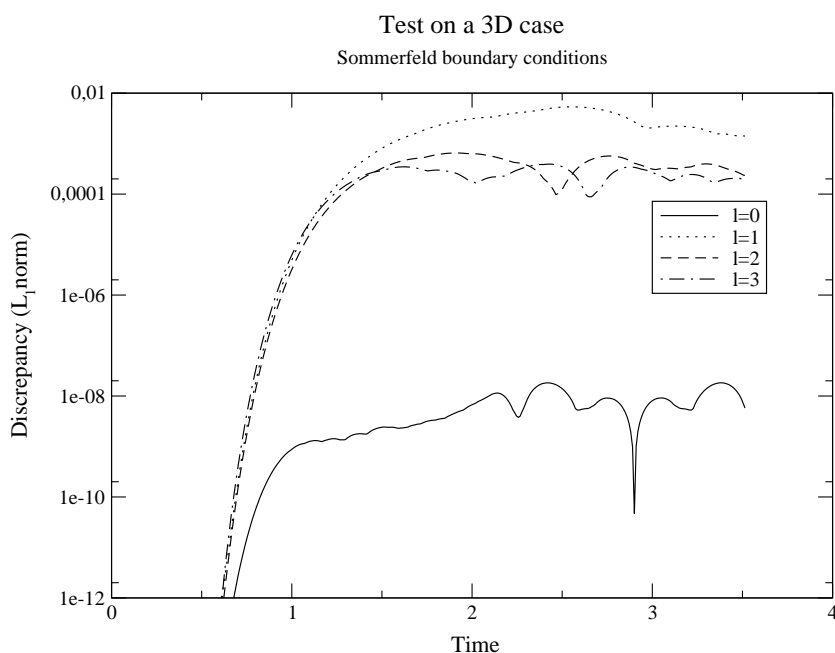


Fig. 2. Time evolution of the first four modes of the wave generated by the source defined in Eqs. (19) and (21); using $B_1\phi = 0$ (4). We took $R = 1.2$, a time-step $\delta t = 10^{-4}$, 33 polynomials for radial decomposition, 17 for θ and 16 for φ .

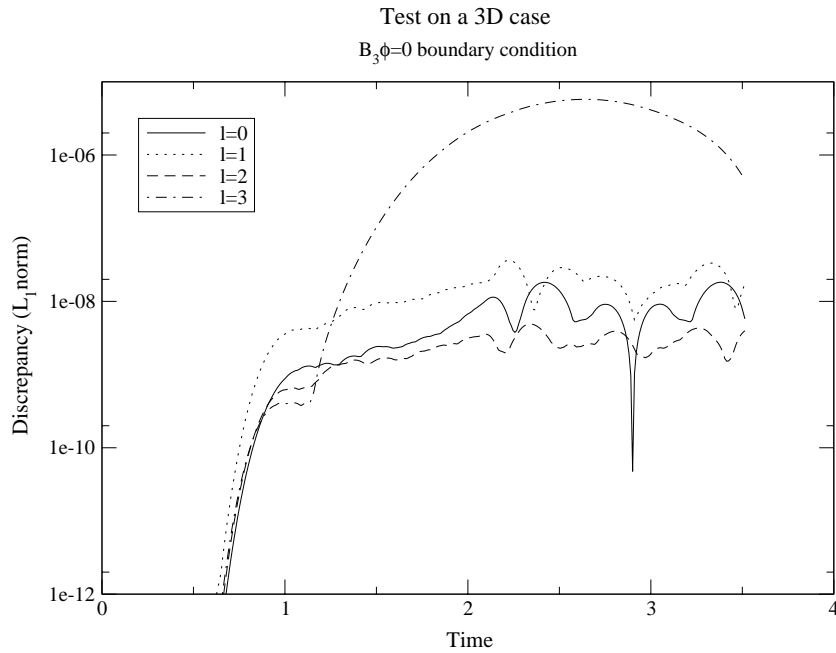


Fig. 3. Time evolution of the first four modes of the wave generated by the source defined in Eqs. (19) and (21); using $B_3\phi = 0$ (16) as the boundary condition. We took $R = 1.2$, a time-step $\delta t = 10^{-4}$, 33 polynomials for radial decomposition, 17 for θ and 16 for φ .

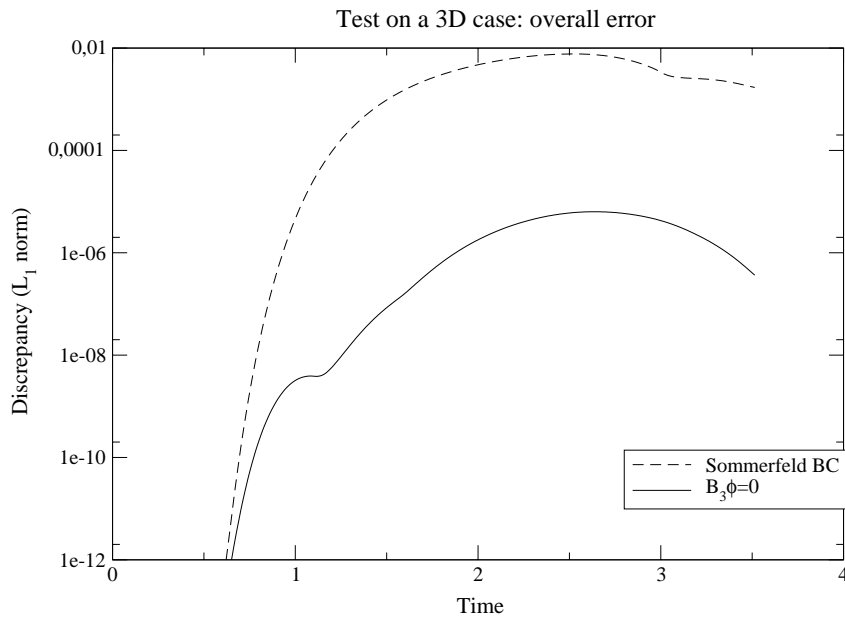


Fig. 4. Time evolution of the error made in the computation of the wave generated by the source defined in Eqs. (19) and (21); using $B_1\phi = 0$ (4) and $B_3\phi = 0$ (16). We took $R = 1.2$, a time-step $\delta t = 10^{-4}$, 33 polynomials for radial decomposition, 17 for θ and 16 for φ .

We have checked this point, namely that the maximal error over the time interval would decrease like $1/R^4$, where R is the distance at which the boundary conditions were imposed. We have also checked that the error decreased both exponentially with the number of coefficients used in r , θ or φ , as one would expect for spectral methods, and like δt^2 (second-order time integration scheme). Fig. 4 shows the overall error as a function of time for both boundary conditions used. Comparing Fig. 4 with Figs. 2 and 3, one can see that most of the error comes from the $l = 1$ term when using $B_1\phi = 0$ boundary condition, and from the $l = 3$ term when using $B_3\phi = 0$. Finally, the computational cost of this enhanced boundary condition is very low with this new approach. For the tests presented here, the difference in CPU time would be of about 10%. This is linked with the fact that our formulation (16) is a perturbation of the Sommerfeld boundary condition (4), where the quantity $\xi_{lm}(t)$ is obtained by simple (ordinary differential equation) integration.

4. Conclusions

The purpose of this paper has been to provide a boundary condition that is well-adapted for the simulation of astrophysical sources of gravitational radiation, whose dominant modes are quadrupolar. We took the series of boundary conditions derived by Bayliss and Turkel [10], truncated at quadrupolar order, and derived a new formulation of that third-order condition in terms of a first-order condition (resembling the classical radiation one), combined with a wave-like equation on the outer boundary of the integration domain. This formulation is simple in the sense that mixed derivatives are (almost) absent.

The numerical implementation using spectral methods and spherical coordinates is straightforward and this formulation of high-order boundary conditions requires only a little more CPU time (less than 10% in our tests) than the simplest first-order condition (4). We have verified that our implementation of this boundary condition had the same efficiency with respect to transparency for dipolar and quadrupolar waves as the Sommerfeld condition (4) for monopolar waves. The precision increases very rapidly (like $1/R^4$) as one imposes the boundary condition further from the source of radiation. These two points are of great interest for the simulation of gravitational radiation from isolated astrophysical sources.

As an alternative, one can cite that more accurate results may be obtained using the so-called 2+2 formalism in the wave zone [18] and matching it to the results in 3+1 formalism² near the source. Our approach is different, much simpler to implement and should give accurate enough results for the Einstein equations.

References

- [1] J. Novak, Review of numerical relativity session, in: R.T. Jantzen, V. Gurzadyan, R. Ruffini (Eds.), Proceedings of the Ninth Marcel Grossman Meeting on General Relativity, Rome, Italy, July 2000, World Scientific, Singapore, 2002.
- [2] J.W. York, Kinematics and dynamics in general relativity, in: L. Smarr (Ed.), Sources of Gravitational Radiation, Cambridge University Press, Cambridge, 1979.
- [3] L. Smarr, J.W. York, Radiation gauge in general relativity, Phys. Rev. D 17 (1978) 1945.
- [4] P. Grandclément, S. Bonazzola, E.ourgoulhon, J.-A. Marck, A multidomain spectral method for scalar and vectorial Poisson equations with noncompact sources, J. Comput. Phys. 170 (2001) 231, doi:10.1006/jcph.2001.6734.
- [5] J. Novak, Spherical neutron star collapse toward a black hole in a tensor-scalar theory of gravity, Phys. Rev. D 57 (1998) 4789.
- [6] S. Frittelli, R. Gomez, Ill-posedness in the Einstein equations, J. Math. Phys. 41 (2000) 5535.
- [7] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativity 5 (2002) 3.
- [8] J.S. Hesthaven, On the analysis and construction of perfectly matched layers for the linearized Euler equations, J. Comput. Phys. 142 (1998) 129.

² This is the *characteristic-Cauchy* matching, CCM.

- [9] J.-L. Lions, J. Méttral, O. Vacus, Well-posed absorbing layer for hyperbolic problems, *Numer. Math.* 92 (2002) 535, doi:10.1007/s002110100263.
- [10] A. Bayliss, E. Turkel, Radiation boundary conditions for wave-like equations, *Comm. Pure Appl. Math.* 33 (1980) 707.
- [11] A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, New York, 1949.
- [12] U.E. Aladl, A.S. Deakin, H. Rasmussen, Nonreflecting boundary condition for the wave equation, *J. Comput. Appl. Math.* 138 (2002) 309.
- [13] B. Engquist, A. Majda, Absorbing boundary conditions for the numerical simulation of waves, *Math. Comput.* 31 (1977) 629.
- [14] D. Gottlieb, S.A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, Society for Industrial and Applied Mathematics, Philadelphia, 1977.
- [15] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zhang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, 1988.
- [16] J.P. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover Publication, Mineola, 2001.
- [17] S. Bonazzola, E.ourgoulhon, J.-A. Marck, Spectral methods in general relativistic astrophysics, *J. Comput. Appl. Math.* 109 (1999) 433.
- [18] R.A. d’Inverno, 2 + 2 formalism and applications, in: J.-A. Marck, J.-P. Lasota (Eds.), *Relativistic Gravitation and Gravitational Radiation*, Cambridge University Press, Cambridge, 1997.